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Exact Energy Eigenvalues for the Doubly Anharmonic Oscillator

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The exact energy eigenvalues for the potential $V(r)=\frac{1}{2}\omega^2r^2+\frac{1}{4}\lambda r^4+\frac{1}{6}\eta r^6$ and the conditions for their occurrence have been determined.

In the present paper we determine the exact energy eigenvalues for the potential

$$V(r)=\omega^2r^2/2+\lambda r^4/4+\eta r^6/6, \qquad \eta>0 \tag{1}$$

This problem has been studied by Leach (1984), Flessas and Das (1980), and Kaushal (1989). The radial part of the Schrödinger equation for the potential (1) can be written as

$$-\frac{1}{2}\frac{d^2u_l}{dr^2}+\left[\left(\frac{1}{2}\omega^2r^2+\frac{1}{4}\lambda r^4+\frac{1}{6}\eta r^6\right)+\frac{l(l+1)}{2r^2}\right]u_l=Eu_l \tag{2}$$

where the symbols have their usual meanings. Now with the substitution

$$u_l(r)=r^{l+1}X_l(r)\exp\left(-\frac{1}{4}\mu r^4-\frac{1}{2}\nu r^2\right) \tag{3}$$

where

$$4\mu\nu=\lambda \tag{4}$$

$$3\mu^2=\eta \tag{5}$$

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Table I. Exact Energy Eigenvalues of the Potential

$$V(r) = \frac{1}{2}\omega^2 r^2 + \frac{1}{4}\lambda r^4 + \frac{1}{6}\eta r^6$$

p	Energy eigenvalues (E)
l	$\beta(l+3/2)$
$l+1$	$\beta(l+5/2)$
$l+2$	$\beta\{(l+5/2) \pm \alpha^3[1+(64/\lambda^2)(l+3/2)]^{1/2}\}$
$l+3$	$\beta\{(l+7/2) \pm \alpha^3[1+(96/\lambda^2)(l+5/2)]^{1/2}\}$
$l+4$	$\beta[(l+7/2)+4\chi^{1/2}\cos(\Phi+2\pi g/3)]$
	$q = 0, 1, 2$
	$\chi = \frac{1}{3}[1+(64/\lambda^2)(l+2)\alpha^3]$
	$\Phi = \tan^{-1}[(27/256)(\lambda^4/\eta^3)\chi^3 - 1]^{1/2}$
$l+5$	$\beta[(l+9/2)+4\chi^{1/2}\cos(\Phi+2\pi g/3)]$
	$q = 0, 1, 2$
	$\chi = \frac{1}{3}[1+(8\alpha^3/\lambda^2)(11l+27)]$
	$\Phi = \tan^{-1}(27\lambda^4\chi^3/16\eta^3(l-3)^2 - 1)^{1/2}$

using equations (3)–(6), we can recast equation (2) as

$$\frac{d^2x_l}{dr^2} - 2\left(\mu r + v r^3 - \frac{l+1}{r}\right)\frac{dX_l}{dr} + (\epsilon + \delta r^2)X_l = 0 \tag{6}$$

with

$$\epsilon = 2E - v(2l+3) \tag{7}$$

$$\delta = v^2 - \omega^2 - \mu(2l+5) \tag{8}$$

Substituting

$$X_l(r) = \sum_{j=0}^{\infty} a_j r^j \tag{9}$$

we get from (4)

$$(j+2)(j+2l+3)c_{j+2} = (2vj - \epsilon)c_j + [2\mu(j-2) - \delta]c_{j-2} \tag{10}$$

Now the energy eigenvalues corresponding to the potential (1) can be found from the condition of the termination of the series (9) by using equation (10), and in this process we get a relation between ω, η, and λ. Now we get a number of eigenvalues corresponding to different relations between ω, η, and λ. In general these relations can be expressed as

$$\frac{3\lambda^2}{16\eta} = \omega^2 + (2p+5)\alpha \tag{11}$$

where

$$\alpha = (\eta/3)^{1/2} \tag{12}$$

$$p = l, l+1, l+2, l+3, l+4, l+5 \tag{13}$$

We have listed the eigenvalues corresponding to the different values of p in Table I, in which we have written

$$\beta = \tfrac{1}{4}\lambda/\alpha \tag{14}$$

It is to be noted that the eigenvalue corresponding to $p = 1$ have been obtained by Flessas and Das (1980) and Kaushal (1989). Kaushal (1989) has incidentally obtained only one eigenvalue. The eigenvalues corresponding to $p = l + 2$ have been obtained by Flessas and Das (1980).

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